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Structure-Preserving Model Reduction Approach for Structured Index-2 Descriptor-Systems

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ABSTRACT

This article presents a balancing-based algorithm for reducing the complexity of structured discrete-time linear time-invariant (DT-LTI) index-2 descriptor systems. The proposed algorithm involves projecting the index-2 system onto a hidden manifold, which converts it into a generalized system. However, this causes the system to lose its sparsity and become dense, which is impractical for large-scale systems. To overcome this issue, the authors enhance the Smith-based iterative method for solving discrete-time algebraic Lyapunov equations, which allow for balanced truncation without explicitly forming the dense system. The proposed algorithm is shown to be efficient and robust through numerical simulations.

1 | Introduction

Discrete-time linear time-invariant (DT-LTI) systems have received remarkable interest in many research studies for their uprising applications in different fields of engineering [1–7]. The models described there are large and complicated in structure since they involve several interconnections. Simulation and optimization of such models require not only large storage but also huge computational cost. Therefore, we are motivated to apply the model order reduction (MOR) [3, 4, 8, 9], which replaces large-scale systems with small-scale systems by preserving the essential properties of the original one.

We assume the DT-LTI descriptor system,

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \quad k \in \mathbb{Z} \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the state, input and output, respectively. The matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Consider the matrix pencil $\lambda E - A$ is regular, i.e., $det(\lambda E - A) \neq 0$, and d-stable, i.e., all its finite eigenvalues of $\lambda E - A$ lie inside the unit circle. Then, $\lambda E - A$ can be represented by a Weierstraß canonical decomposition [5],

$$\mathbf{S}(\lambda E - A)\mathbf{T} = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \tag{2}$$

where $\mathbf{S}$ and $\mathbf{T}$ are nonsingular, J corresponds to the finite eigenvalues of $\lambda E - A$ (including zero eigenvalues), and the nilpotent N corresponds to the infinite eigenvalues of $\lambda E - A$. Then we define the index ν of the pencil $\lambda E - A$ or the index of system (2) as $N^{\nu-1} \neq 0$ and $N^{\nu} = 0$ [10, 11].

In our case matrix E to be singular for the descriptor system. This research is focused on the DT-LTI index-2 descriptor system of the following form,

$$\underbrace{\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}}_{x(k)} + \underbrace{\begin{bmatrix} B_1 \\ 0 \end{bmatrix}}_{B} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} C_1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{3}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$ $(n_1 > n_2, n_1 + n_2 = n)$ are the states, $E_{11} \in \mathbb{R}^{n_1 \times n_1}$ is symmetric positive definite, $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ is of full rank and $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m}$ and $C_1 \in \mathbb{R}^{p \times n_1}$. We assume that A_{12} has full column rank, i.e., $\mathrm{rank}(A_{12}) = n_2$. Structured discrete-time systems arise in digital control systems and sample data analysis [12, 13].

As a model problem, we can consider a spatial discretization of the linearised Stokes equations around a steady state [14]. The finite difference method can be used to discretize the spatial derivatives. As a result, one receives an index-2 model problem like (1) with a specific structure as in (3).

Discrete-time systems also arise from the continuous-time state equations in the process of numerical approximation, such as from the spatial discretization of Navier-Stokes, or Oseen equations [8, 15–17]. These equations are widely used in computational fluid dynamics and engineering applications to describe various problems.

The basic idea of MOR is to derive a smaller model from a higher-order model where the important properties of the system such as the stability and the passivity of the higher-order model are preserved in the smaller model. Apart from these, the smaller model will produce a negligible approximation error. For many researchers in the last few decades, MOR techniques have become an extensive topic to the state of the art research [6, 7, 12, 16–20]. One of the popular MOR methods is balanced truncation (BT) which has been proposed in several research articles in the last few decades, see [4, 20] and the references therein. Also, a balanced truncation reduction strategy for index-1 descriptor systems, both for continuous and discrete settings, became popular in the last two decades, and it is a popular choice for many engineering applications [21, 22]. There are a few existing works that propose balanced truncation for model reduction of index-2 LTI systems can be found in [16–18, 23]. However, those works focused on the continuous-time setting of index-2 descriptor systems. It is to be noted that the index-2 descriptor system with the specific sparse structure has not been treated before in the case of discrete-time setting. Therefore, there is a lack of research on that particular model, especially on model reduction. This work builds upon the author's previous work [24] but provides a more comprehensive analysis and details. Numerical results from a large-dimensional mathematical model are also provided in this paper. The current study aims to present an effective strategy for model order reduction of a discrete-time index-2 descriptor system.

The BT-based MOR of continuous-time large-scale descriptor systems was developed first by Stykel in [22]. In principle, there the proposed method was based on splitting the descriptor system into proper and improper subsystems corresponding to the deflating subspaces of the associated matrix pencil to finite and infinite eigenvalues and then reducing only the order of the proper subsystem. This approach requires the availability of spectral projectors onto the respective subspaces. Recently, a method for BT of structured large-scale descriptor systems of index-2 has been developed [16] that avoids the computation of spectral projectors explicitly. Instead, it implicitly performs an index reduction by projection to the inherent or hidden manifold on which the solution evolves. In [16], the authors presented the MOR strategy of continuous-time LTI system only.

In this paper, we present an efficient technique for the MOR of index-2 discrete-time differential algebraic equations without using a spectral projector explicitly. Preliminary results of this article were presented and published in [24]. In comparison to [24], here we present implementation details, including proof of some necessary theorems. The main computational bottleneck in BT methods is to solve two Lyapunov equations associated with the system. To implement the BT method of the index-2 system, we need to solve two projected discrete-time algebraic Lyapunov equations (DALEs). This paper also discusses how to solve them efficiently by using the Smith method to retain the sparsity of the system.

The paper is organized as follows: In Section 2, the standard balanced truncation MOR method for generalized systems and the Smith iterative method for solving generalized DALEs are reviewed. Section 3 explains how to reshape the system (3) into a generalized system. Section 4 shows how the system can be reduced via balanced truncation without being converted into a non-singular system. In Section 5, a unique method based on the Smith iterative method is presented for approximating the solutions of the corresponding Lyapunov equations for the projected non-singular system, without explicitly computing the projectors. In Section 6, the proposed algorithm's performance and robustness are tested, and the concluding remarks are given in Section 7.

2 | Preliminaries

Before describing the balanced truncation model order reduction (BT-MOR) for an index-2 descriptor system, we give a brief review of the basics of standard balanced truncation for generalized systems and the standard procedure of the Smith iterative method for the approximation of the Cholesky factors of the Gramians.

2.1 | Model Reduction by Balanced Truncation

Let us consider a generalized discrete-time linear time-invariant system

$$\begin{aligned} \mathcal{E}\mathcal{X}(k+1) &= \mathcal{A}\mathcal{X}(k) + \mathcal{B}\mathcal{U}(k) \\ \mathcal{Y}(k) &= \mathcal{C}\mathcal{X}(k), \quad k \in \mathbb{Z} \end{aligned} \tag{4}$$

where $\mathcal{E} \in \mathbb{R}^{\mathfrak{n} \times \mathfrak{n}}$ is a non-singular matrix, and $\mathcal{A} \in \mathbb{R}^{\mathfrak{n} \times \mathfrak{n}}$, $\mathcal{B} \in \mathbb{R}^{\mathfrak{n} \times \mathfrak{m}}$ and $\mathcal{C} \in \mathbb{R}^{\mathfrak{p} \times \mathfrak{n}}$. Using BT-MOR method, a reduced-order model of dimension $\mathfrak{r}$ can be computed for the system (4) as

$$\begin{aligned}\hat{\mathcal{E}}\hat{\mathcal{X}}(k+1) &= \hat{\mathcal{A}}\hat{\mathcal{X}}(k) + \hat{\mathcal{B}}\mathcal{U}(k)\\ \hat{\mathcal{Y}}(k) &= \hat{\mathcal{C}}\hat{\mathcal{X}}(k)\end{aligned} \tag{5}$$

where $\hat{\mathcal{E}}, \hat{\mathcal{A}} \in \mathbb{R}^{\mathfrak{r}\times\mathfrak{r}}$, $\hat{\mathcal{B}} \in \mathbb{R}^{\mathfrak{r}\times\mathfrak{m}}$ and $\hat{\mathcal{C}} \in \mathbb{R}^{\mathfrak{p}\times\mathfrak{r}}$ and $\mathfrak{r} \ll \mathfrak{n}$.

The inevitable part of the BT-MOR method is to solve the DALEs associated to system (4), which are given by

$$\mathcal{E}\mathcal{P}\mathcal{E}^T - \mathcal{A}\mathcal{P}\mathcal{A}^T = \mathcal{B}\mathcal{B}^T \tag{6a}$$

$$\mathcal{E}^T\mathcal{Q}\mathcal{E} - \mathcal{A}^T\mathcal{Q}\mathcal{A} = \mathcal{C}^T\mathcal{C} \tag{6b}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the unique positive semidefinite solutions of (6a) and (6b), respectively [25]. Note that $\mathcal{P}$ and $\mathcal{E}^T\mathcal{Q}\mathcal{E}$ are known as the controllability and observability Gramians of system (4).

The system (4) is known to be asymptotically stable if all finite eigenvalues of the matrix pair $(\mathcal{E}, \mathcal{A})$ are located inside the unit disk [26]. For the stable matrix pair $(\mathcal{E}, \mathcal{A})$ and the Cholesky factors $\mathcal{R}$ and $\mathcal{L}$ satisfy $\mathcal{P} = \mathcal{R}\mathcal{R}^T$ and $\mathcal{Q} = \mathcal{L}\mathcal{L}^T$, we define the Hankel singular values (HSVs) of (4) through the singular value decomposition (SVD)

$$\mathcal{R}^T\mathcal{E}^T\mathcal{L} = [\mathbf{U}, \mathbf{S}, \mathbf{V}^T] \tag{7}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, and $\mathbf{S} = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ contains the HSVs σ_j, $j = 1, 2, \dots, n$, of system (4). The HSVs are another fundamental component of the BT, which helps us to identify the states that are laborious to control and arduous to observe. In simple words, the HSVs that have a small magnitude correspond to these disruptive states [3], which means states which are difficult to reach and difficult to observe. Hence, by removing these states, a smaller system through a balanced realization of the original system can be attained.

The SVD in (7) can be computed as

$$\mathcal{R}^T\ \mathcal{E}^T\ \mathcal{L} = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}\begin{bmatrix}\mathbf{S}_1 & \\ & \mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix}^T \tag{8}$$

where $\begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}$ and $\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix}^T$ are orthogonal, $\mathbf{S}_1 = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$ is nonsingular, $\mathbf{S}_2 = \mathrm{diag}(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n)$, and $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} \geq \cdots \geq \sigma_n$.

Now, the truncation matrices can be constructed as follows

$$\mathcal{T}_L = \mathcal{R}\mathbf{U}_1\mathbf{S}_1^{-\frac{1}{2}}, \quad \mathcal{T}_R = \mathcal{L}\mathbf{V}_1\mathbf{S}_1^{-\frac{1}{2}} \tag{9}$$

Then, these truncation matrices can be applied to the original system to get a reduced order system as

$$\hat{\mathcal{E}} = \mathcal{T}_L^T\mathcal{E}\mathcal{T}_R, \quad \hat{\mathcal{A}} = \mathcal{T}_L^T\mathcal{A}\mathcal{T}_R, \quad \hat{\mathcal{B}} = \mathcal{T}_L^T\mathcal{B}, \quad \hat{\mathcal{C}} = \mathcal{C}\mathcal{T}_R \tag{10}$$

To verify how close the reduced order system approximates the original system, the $\mathcal{H}_\infty$ norm error bound can be utilized as given below,

$$||\mathcal{H} - \hat{\mathcal{H}}||_{\mathcal{H}_\infty} = \sup ||\mathcal{H}(j\omega) - \hat{\mathcal{H}}(j\omega)||_2 \leq 2\ \mathrm{trace}\ (\mathbf{S}_2) \tag{11}$$

ALGORITHM 1 | Smith method for (6a).

Input: $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$.
Output: Low Rank Cholesky factor$\mathcal{R}_i$; such that $\mathcal{P} \approx \mathcal{R}_i\mathcal{R}_i^T$.

$\mathcal{W}_1 = \mathcal{E}^{-1}\mathcal{B}$
$\mathcal{R}_1 = \mathcal{W}_1$
for $i = 2, 3, \dots,$ **do**
  $\mathcal{W}_i = \left(\mathcal{E}^{-1}\mathcal{A}\right)\mathcal{W}_{i-1}$
  $\mathcal{R}_i = [\mathcal{R}_{i-1} \quad \mathcal{W}_i]$
end for

where $\mathcal{H} = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ and $\hat{\mathcal{H}} = \hat{\mathcal{C}}(s\hat{\mathcal{E}} - \hat{\mathcal{A}})^{-1}\hat{\mathcal{B}}$ are the transfer functions of the original and reduced order systems, respectively.

2.2 | Lyapunov Equations and Their Solutions

Lyapunov equations are omnipresent in several areas of control and engineering. Since these equations are often encountered in the encompassing scope of applications, plentiful methods have been dedicated to finding solutions to Lyapunov equations [11, 19, 27–30]. We consider the Smith method [11, 29] for solving the DALEs (6a) and (6b) iteratively.

Consider equation (6a) (the observability Lyapunov equation (6b) can be treated similarly) and multiplying the left-hand and the right-hand side of (6a) by $\mathcal{E}^{-1}$ and $\mathcal{E}^{-T}$ respectively, we get

$$\mathcal{P} - (\mathcal{E}^{-1}\mathcal{A})\mathcal{P}(\mathcal{E}^{-1}\mathcal{A})^T = (\mathcal{E}^{-1}\mathcal{B})(\mathcal{E}^{-1}\mathcal{B})^T \tag{12}$$

The solution of $\mathcal{P}$ in (12) can be estimated by utilizing the following Smith iterations [28, 29]

$$\mathcal{P}_i = \sum_{l=0}^{i-1}(\mathcal{E}^{-1}\mathcal{A})^l(\mathcal{E}^{-1}\mathcal{B})(\mathcal{E}^{-1}\mathcal{B})^T\left((\mathcal{E}^{-1}\mathcal{A})^T\right)^l \tag{13}$$

However, instead of estimating the solution $\mathcal{P}$, we estimate its Cholesky factor $\mathcal{R}$ such that $\mathcal{P}_i \approx \mathcal{R}_i\mathcal{R}_i^T$ and $\mathcal{P} = \mathcal{R}\mathcal{R}^T \approx \mathcal{R}_i\mathcal{R}_i^T$ is achieved after the successful i iterations. The Cholesky factor $\mathcal{R}_i$ can be specified as

$$\mathcal{R}_i = \left[\mathcal{E}^{-1}\mathcal{B}, (\mathcal{E}^{-1}\mathcal{A})(\mathcal{E}^{-1}\mathcal{B}), \dots \dots, (\mathcal{E}^{-1}\mathcal{A})^{i-1}(\mathcal{E}^{-1}\mathcal{B})\right] \tag{14}$$

The whole iterative process of solving (6a) using the Smith method is summarized in Algorithm 1.

For an effective iteration process, it is very important to observe the convergence history of the approximate solution toward the exact solution. It is suggested to stop the iterations in Algorithm 1 as soon as the *normalized residual norm*, defined as

$$\eta(\mathcal{R}_i) = \frac{\|\mathcal{E}\mathcal{R}_i\mathcal{R}_i^T\mathcal{E}^T - \mathcal{A}\mathcal{R}_i\mathcal{R}_i^T\mathcal{A}^T - \mathcal{B}\mathcal{B}^T\|_F}{\|\mathcal{B}\mathcal{B}^T\|_F} \tag{15}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$, where tol is a user-defined tolerance. When the number of columns in $\mathcal{B}$ is big, $\mathcal{R}_i$ will grow fast and it may face rank deficiency. This is because, in each iteration step, Algorithm 1 will add as many columns as there are in $\mathcal{B}$ to the previous $\mathcal{R}_i$. Hence, we truncate those redundant columns of $\mathcal{R}_i$. Consider that $\mathcal{R}_i \in \mathbb{R}^{\mathfrak{n}\times\mathfrak{r}_c}$ has the numerical rank $\text{rank}(\mathcal{R}_i, \tau) = \mathfrak{r}_n < \mathfrak{r}_c$ with a prescribed tolerance τ. Then we propose to use the rank-revealing QR decomposition (RRQR) [4]

$$\mathcal{R}_i^T = \overline{Q}_i \mathcal{Z}_i \overline{\Pi}_i^T, \qquad \overline{Q}_i = \begin{bmatrix} \overline{Q}_{1,i}, & \overline{Q}_{2,i} \end{bmatrix}, \quad \mathcal{Z}_i = \begin{bmatrix} \mathcal{Z}_{i,11} & \mathcal{Z}_{i,12} \\ 0 & \mathcal{Z}_{i,22} \end{bmatrix}$$

where $\overline{Q}_i$ is orthogonal, $\overline{\Pi}_i$ is a permutation matrix, $\mathcal{Z}_{i,11} \in \mathbb{R}^{\mathfrak{r}_n\times\mathfrak{r}_n}$ is upper triangular and $\|\mathcal{Z}_{i,22}\|_F \leq \tau$. Setting $\mathcal{Z}_{i,22} \approx 0$ and $\overline{\mathcal{R}}_i^T = \overline{Q}_{1,i} \, [\mathcal{Z}_{i,11} \mathcal{Z}_{i,12}] \, \Pi_i^T$, we find that $\overline{\mathcal{R}}_i \overline{\mathcal{R}}_i^T \approx \mathcal{P}$. Note that we do not need to compute $\overline{Q}_{1,i}$, since this matrix cancels out in the product $\overline{\mathcal{R}}_i \overline{\mathcal{R}}_i^T$ because of its orthogonal property.

Remark 1. The observability Gramian for system (4) can also be determined from the corresponding DALEs (6b). In that case, we estimate the Cholesky factor $\mathcal{L}$ of the solution $\mathcal{Q}$ such that $\mathcal{Q} \approx \mathcal{L}_i \mathcal{L}_i^T$. For this estimation, we replace $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$ of Algorithm 1 by $\mathcal{E}^T$, $\mathcal{A}^T$, and $\mathcal{C}^T$, respectively.

3 | Reformulate the Index-2 Descriptor System

The goal of this paper is to transfer the concepts and results presented in [16] into the discrete-time case. We derive the method of converting an index-2 discrete-time descriptor system into a non-singular system by manipulating the structure of the system's equations. The algebraic equations extracted from the matrix equations (3) have the following forms:

$$E_{11} x_1(k+1) = A_{11} x_1(k) + A_{12} x_2(k) + B_1 u(k) \tag{16a}$$

$$0 = A_{12}^T x_1(k) \tag{16b}$$

$$y(k) = C_1 x_1(k) \tag{16c}$$

The structure of Equations (16a) and (16b) allows us to express system (3) with a difference equation for $x_1(k)$ independent of $x_2(k)$, and an algebraic equation of $x_2(k)$ portrayed as a function of $x_1(k)$. Applying (16b) into (16a), one can find [24]

$$\begin{aligned} x_2(k) = &-(A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} A_{11} x_1(k) \\ &- (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} B_1 u(k) \end{aligned} \tag{17}$$

Incorporating (17) into (16a), we can redefine (16a) as

$$E_{11} x_1(k+1) = \Pi A_{11} x_1(k) + \Pi B_1 u(k) \tag{18}$$

where

$$\Pi = I - A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} \tag{19}$$

is an oblique projector. It holds some important properties such as $\Pi^2 = \Pi$, $\Pi E_{11} = E_{11} \Pi^T$, $\text{null}(\Pi) = \text{range}(A_{12})$, and $\text{range}(\Pi) = \text{null}(A_{12}^T E_{11}^{-1})$ [16]. Another implication of these properties is that

$$A_{12}^T z = 0 \quad \text{if and only if} \quad \Pi^T z = z \tag{20}$$

Hence, it is evident from equations (16b) and (20) that $\Pi^T x_1(k) = x_1(k)$. Now, using these identities and pre-multiplying (18) by Π, the descriptor system (3) can be expressed as

$$\begin{aligned} \Pi E_{11} \Pi^T x_1(k+1) &= \Pi A_{11} \Pi^T x_1(k) + \Pi B_1 u(k) \\ y(k) &= C_1 \Pi^T x_1(k) \end{aligned} \tag{21}$$

The system dynamics of (21) are projected onto the $(n_1 - n_2)$ dimensional subspace null(Π). However, this subspace is still represented in the coordinates of the n_1 dimensional space. The $(n_1 - n_2)$ dimensional representation of (21) can be made explicit by exploiting the *thin* SVD [4]

$$\Pi = \bar{U} \overline{\Sigma} \overline{V}^T = \begin{bmatrix} \bar{U}_c & \bar{U}_p \end{bmatrix} \begin{bmatrix} \overline{\Sigma}_c & \\ & 0 \end{bmatrix} \begin{bmatrix} \overline{V}_c^T \\ \overline{V}_p^T \end{bmatrix} = \bar{U}_c \, \overline{\Sigma}_c \, \overline{V}_c^T = \Phi \Theta^T \tag{22}$$

where $\Phi = \bar{U}_c$, $\Theta = \overline{V}_c \in \mathbb{R}^{n_1 \times (n_1 - n_2)}$, and $\bar{U}_c$, $\overline{V}_c$ contain the leading $n_1 - n_2$ columns of $\bar{U}, \overline{V} \in \mathbb{R}^{n_1 \times n_1}$, respectively. In (22), $\overline{\Sigma}_c$ is an identity matrix as Π is a projector; see section 4 in [23]. Moreover, Φ, Θ satisfy,

$$\Phi^T \Theta = I \tag{23}$$

Using the notation $\tilde{x}(k) = \Phi^T x_1(k)$, we can rewrite (21) as,

$$\begin{aligned} \Theta^T E_{11} \Theta \tilde{x}(k+1) &= \Theta^T A_{11} \Theta \tilde{x}(k) + \Theta^T B_1 u(k) \\ y(k) &= C_1 \Theta \tilde{x}(k) \end{aligned} \tag{24}$$

The system expressed by the equations in (24) can be classified as a generalized discrete-time system due to the fact that the system's matrices are non-singular. Therefore, standard techniques for model reduction by BT can be applied to it.

4 | Model Reduction Without Reformulation

In the previous section, we showed how an index-2 descriptor system can be reformulated into a non-singular generalized system for which the standard BT-MOR can be applied. Now in this section, we derive another approach where it is not necessary to form (24) explicitly for this BT-MOR method to be applied. This is important since forming (24) is often impossible for large-scale systems. Now, the corresponding Lyapunov equations for the system (24) are

$$\Theta^T E_{11} \Theta \tilde{P} \Theta^T E_{11} \Theta - \Theta^T A_{11} \Theta \tilde{P} \Theta^T A_{11}^T \Theta = \Theta^T B_1 B_1^T \Theta \tag{25a}$$

$$\Theta^T E_{11} \Theta \tilde{Q} \Theta^T E_{11} \Theta - \Theta^T A_{11}^T \Theta \tilde{Q} \Theta^T A_{11} \Theta = \Theta^T C_1^T C_1 \Theta \tag{25b}$$

where the solutions $\tilde{P}$ and $\tilde{Q}$ are unique and positive (semi-) definite [25].

Now we multiply both (25a) and (25b) by Φ and Φ^T from the left and the right, accordingly, and we obtain

$$\Pi E_{11} \Pi^T P \Pi E_{11} \Pi^T - \Pi A_{11} \Pi^T P \Pi A_{11}^T \Pi^T = \Pi B_1 B_1^T \Pi^T \tag{26a}$$

$$\Pi E_{11} \Pi^T Q \Pi E_{11} \Pi^T - \Pi A_{11}^T \Pi^T Q \Pi A_{11} \Pi^T = \Pi C_1^T C_1 \Pi^T \tag{26b}$$

where $P = \Theta\tilde{P}\Theta^T \in R^{n_1\times n_1}$ and $Q = \Theta\tilde{Q}\Theta^T \in R^{n_1\times n_1}$ are unique and $P = \Pi^T P\Pi$, $Q = \Pi^T Q\Pi$. Note that in the above manipulations, we use the properties $\Theta = \Pi^T\Theta$ (due to (20)) and (23). The system matrices in (26a) and (26b) are singular since Π is singular.

Let us assume that we approximate the low-rank Cholesky factors as $P = RR^T$ and $Q = LL^T$. Also, $\tilde{P} = \tilde{R}\tilde{R}^T$ and $\tilde{Q} = \tilde{L}\tilde{L}^T$ in (25a) and (25b). Then, these factors are related by the following relations

$$R = \Theta\tilde{R}, \quad L = \Theta\tilde{L} \tag{27}$$

For large-scale systems, it is almost impossible to compute Θ because of high computational costs and memory limitations. Furthermore, Θ will destroy the structures of the system matrices when multiplied with them. Therefore, a direct application of Θ to the MOR techniques of system (21) or (1) is not anticipated.

The truncation matrices for model reduction of the system (24) can be constructed as

$$\tilde{T}_L = \tilde{R}U_1\Sigma_1^{-\frac{1}{2}}, \quad \tilde{T}_R = \tilde{L}V_1\Sigma_1^{-\frac{1}{2}} \tag{28}$$

where $U_1, V_1 \in \mathbb{R}^{(n_1-n_2)\times l}$ are the leading l columns of $U, V \in \mathbb{R}^{(n_1-n_2)\times(n_1-n_2)}$, and $\Sigma_1 \in \mathbb{R}^{l\times l}$ is the leading upper block of the diagonal Σ in the SVD

$$\tilde{R}^T\Theta^T E_{11}\Theta\tilde{L} = \tilde{R}^T\tilde{E}\tilde{L} = U\Sigma V^T$$

Furthermore, we also find that

$$R^T\Pi E_{11}\Pi^T L = \tilde{R}^T\Theta^T E_{11}\Theta\tilde{L} = \tilde{R}^T\tilde{E}\tilde{L} = U\Sigma V^T$$

Therefore, we can generate the truncation matrices for model reduction of system (21) as

$$T_L = RU_1\Sigma_1^{-\frac{1}{2}}, \quad T_R = LV_1\Sigma_1^{-\frac{1}{2}} \tag{29}$$

Further observations reveal that

$$T_L = RU_1\Sigma_1^{-\frac{1}{2}} = \Theta\tilde{R}U_1\Sigma_1^{-\frac{1}{2}} = \Theta\tilde{T}_L = \Theta\Phi^T\Theta\tilde{T}_L = \Pi^T T_L \tag{30a}$$

$$T_R = LV_1\Sigma_1^{-\frac{1}{2}} = \Theta\tilde{L}V_1\Sigma_1^{-\frac{1}{2}} = \Theta\tilde{T}_R = \Theta\Phi^T\Theta\tilde{T}_R = \Pi^T T_R \tag{30b}$$

Now, the matrices of the reduced order model can be obtained by applying the transformations T_L and T_R on (21) as

$$\begin{aligned} &\hat{E} = T_L^T\Pi E_{11}\Pi^T T_R, \quad \hat{A} = T_L^T\Pi A_{11}\Pi^T T_R \\ &\hat{B} = T_L^T\Pi B_1, \quad \hat{C} = C_1\Pi^T T_R \end{aligned} \tag{31}$$

The properties shown in (30) allow us to get rid of Π from the transformations of (31) to find

$$\hat{E} = T_L^T E_{11}T_R, \quad \hat{A} = T_L^T A_{11}T_R, \quad \hat{B} = T_L^T B_1, \quad \hat{C} = C_1 T_R \tag{32}$$

Hence, the desired reduced order system has dimension r and it can be attained as

$$\begin{aligned} \hat{E}\hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) \\ \hat{y}(k) &= \hat{C}(k)\hat{x}(k) \end{aligned} \tag{33}$$

5 | Projector-Free Solutions of Dales

In the previous section, we showed that we can acquire the reduced order model (33) without reformulating the original descriptor system (3) into a projected non-singular system (21). In this section, we modify the Smith iterative method with the goal of computing the Cholesky factors R and L of the Gramians without explicit use of Π.

As a first step of solving equation (26), one might be tempted to multiply the equation by Π^{-1}. Unfortunately, this approach is invalid since the matrices surrounding Π are singular and hence cannot be inverted. However, they can be made invertible by restricting to the subspace.

Let us define

$$\tilde{E} = \Pi E_{11}\Pi^T, \quad \tilde{A} = \Pi A_{11}\Pi^T, \quad \tilde{B} = \Pi B_1, \quad \tilde{C} = C_1\Pi^T \tag{34}$$

With these notations, (26) can be written as

$$\tilde{E}P\tilde{E}^T - \tilde{A}P\tilde{A}^T = \tilde{B}\tilde{B}^T \tag{35a}$$

$$\tilde{E}^T Q\tilde{E} - \tilde{A}^T Q\tilde{A} = \tilde{C}^T\tilde{C} \tag{35b}$$

Lemma 1. *Consider Θ in (23) such that $\Theta^T E_1\Theta$ is invertible. The matrix*

$$\tilde{E}^I = \Theta(\Theta^T E_{11}\Theta)^{-1}\Theta^T \tag{36}$$

satisfies $\tilde{E}^I\tilde{E} = \Pi^T$ and $\tilde{E}\tilde{E}^I = \Pi$

Proof.

$$\begin{aligned} \tilde{E}^I\tilde{E} &= \Theta(\Theta^T E_{11}\Theta)^{-1}\Theta^T\Pi E_{11}\Pi^T \\ &= \Theta(\Theta^T E_{11}\Theta)^{-1}\Theta^T\Phi\Theta^T E_{11}\Theta\Phi^T \\ &= \Theta(\Theta^T E_1\Theta)^{-1}(\Theta^T E_{11}\Theta)\Phi^T \\ &= \Theta\Phi^T \\ &= \Pi^T \end{aligned}$$

□

Remark 2. The other identity can be proven similarly.

From (35a), using relation $P = \Pi^T P\Pi$ and Lemma 1, it is evident that the Gramian P satisfies the following projected Stein equation

$$P = \tilde{E}^I\tilde{B}\tilde{B}^T\tilde{E}^I + \tilde{E}^I\tilde{A}P\tilde{A}^T\tilde{E}^I \tag{37}$$

Equation (37) is the foundation of numerous numerical methods for approximating the Gramians in large-scale problems [4, 19, 28–31]. However, the straightforward application of any of these approaches requires the evaluation of $\tilde{A}$ and $\tilde{E}^I$, which eventually implies the requirement of Π and computation of Θ. The computations of these projectors are not desirable. Therefore, we propose an algorithm based on Smith iterative method [28, 29] which approximates the solution of (37) without requiring the application and computation of Π or Θ explicitly.

Lemma 2. ([16]). *The matrix Z satisfies $Z = \Pi^T Z$ and $E_{11} Z = \Pi F$ if and only if*

$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{38}$$

Proof. The proof can be found in [16]. □

Lemma 3. ([16]). *If $M = \Pi^T M$, then we can compute*

$$Z = \tilde{E}^I \tilde{A} M \tag{39}$$

using the following two steps:

1. $F = A_{11} M$.

2. Solve the block matrix equation

$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{40}$$

Proof. The proof can be found in [16]. □

Lemma 4. *The matrix*

$$Z = \tilde{E}^I \tilde{B} \tag{41}$$

solves

$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{42}$$

Proof. Lemma 3 implies that the matrix Z can be obtained by solving (42) which satisfy relations $Z = \Pi^T Z$ and $\tilde{E} Z = \Pi B_1 = \tilde{B}$. Now Lemma 1 and $Z = \Pi^T Z$ can be used to deduce (41). □

Algorithm 2 summarizes all the above-mentioned modified steps of Smith iterative process to solve (26a).

The stopping criteria of Algorithm 2 can be defined in a similar way of (15). Algorithm 2 can be continued as long as the normalized residual norm

$$\eta(R_i) = \frac{\| \tilde{E} R_i R_i^T \tilde{E}^T - \tilde{A} R_i R_i^T \tilde{A}^T - \tilde{B} \tilde{B}^T \|_F}{\| \tilde{B} \tilde{B}^T \|_F} \tag{43}$$

satisfies the condition $\eta(R_i) < tol$ for a pre-defined tolerance tol.

Remark 3. To solve observability Lyapunov equation (26b), replace B_1 with C_1^T in step 1, and A_{11} with A_{11}^T in step 4 of Algorithm 2.

6 | Numerical Results

In this section, we illustrate the simulation results from two different model problems. The MATLAB codes for the reduced order models and the respective log files can be found at https://github.com/SahadetNSU/MOR-INDX2.

ALGORITHM 2 | Structure-exploiting Smith iterative method.

Input: A_{11}, A_{12}, E_{11}, B_1.
Output: R_i, such that $P = RR^T \approx R_i R_i^T$.

1: Solve
$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ \Lambda \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$
2: $R_1 = Z_1$
3: **for** $i = 2, 3, 4 \ldots$ **do**
4: $\quad Z_{i-1} = A_{11} Z_{i-1}$
5: $\quad$ Solve
$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z_i \\ \Lambda \end{bmatrix} = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$
6: $\quad R_i = [R_{i-1} \quad Z_i]$
7: **end for**

6.1 | Example-1: Artificial Data

To corroborate the efficiency of our proposed algorithm, we used artificial data which is constructed in such a way that it preserves the structure criteria of an index-2 descriptor system. The computations are carried out using MATLAB 9.4.0.813654 (R2018a) on a board with an Intel(R) Core (TM) i7-8700 CPU of 3.20-GHz clock speed and 32GB RAM running Microsoft Windows 10.

We consider the dimension of the original system to be $n = 20$. Here, $m = 2$, and $p = 3$. Matrices A and E have dimension 20×20, while, B and C are bearing 20×2 and 3×20 as dimensions, respectively. For this model, we consider $n_1 = 18$, whereas $n_2 = 2$. Here, A_{11} and E_{11} have the dimension 18×18, and the B_1 and C_1 have the dimension 18×2 and 3×18, respectively. The constructions of the system matrices are given below. We consider $A_{21} = A_{12}^T$ in this model formulation.

In this reformulation, we also consider,

$$c_1 = 0.1, \quad c_2 = 0.6, \quad c_3 = 0.9$$
$$s_1 = 0.5, \quad s_2 = 0.3, \quad s_3 = 0.8$$
$$t_1 = 0.7, \quad t_2 = 0.8, \quad t_3 = 0.9$$

We formulate $E_{11} = \mathrm{eye}(n_1, n_1)$. Then,

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

Also,

$$A_{1a} = 0.1 * \begin{bmatrix} -c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 & -s_3 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_3 & t_1 & t_2 \\ s_1 & -s_2 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t_2 & t_3 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & -c_1 & c_2 \\ t_1 & t_2 & -t_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & c_3 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -s_3 & s_1 & -s_2 \end{bmatrix}$$

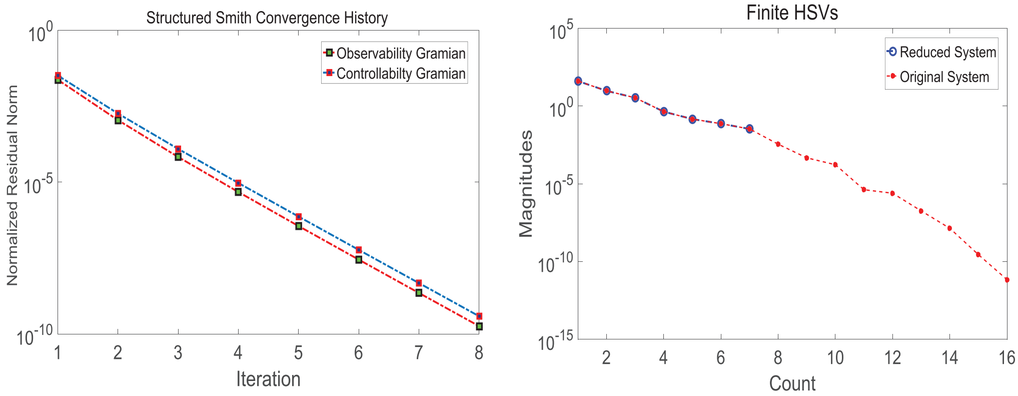

FIGURE 1 | (a) Normalized residual norms in structure-exploiting Smith iteration (left), (b) Hankel singular values (HSVs) of original and reduced systems (right).

and formulate,

$$A_{11} = \begin{bmatrix} A_{la} & A_{la}^T \\ A_{la}^T & A_{la} \end{bmatrix}$$

We construct

$$A_{lb} = \begin{bmatrix} c_1 & -s_2 & -t_3 & s_1 & -t_2 & -c3 & t_1 & -c_2 & -s_3 \\ -s_3 & t_1 & c_2 & s_2 & -t_3 & -c1 & -s_3 & t_1 & c_2 \end{bmatrix}^T$$

$A_{12} = \begin{bmatrix} A_{lb} \\ A_{lb} \end{bmatrix}$, $A_{21} = A_{12}^T$, $A_{22} = \text{zeros}(n - n_1, n - n_1)$.

Then,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

We also construct the input and the output matrices as

$$B_1 = \begin{bmatrix} 4 & -1 & s_3+1 & 1 & 0 & -2 & 0 & 1 & s_1+1 & -s_1+2 & 1 & 0 & s_1+1 & -2 & 1 & -1 & 0 & -l_3 \\ 1 & -1 & s_3 & 1 & 0 & -2 & 0 & 1 & s_1 & 2 & 1 & -1 & s_2 & -2 & 1 & 0 & & l_3 \end{bmatrix}^T$$

$B_2 = \text{zeros}(n - n_1, m), \quad B = [B_1^T, \quad B_2^T]^T.$

Also,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & s_1+1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & s_2+1 & -s_1+2 & \\ 0 & 0 & 1 & 0 & 0 & .05+c_1 & s_3+1 & -s_2+2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & -s_1 & -s_3-1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -s_2+1 & -s_1-2 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & .05+c_1 & -s_3 & -s_2 \end{bmatrix}$$

$C_2 = \text{zeros}(p, n - n_1), \quad C = [C_1, \quad C_2].$

The reduced order system (33) has dimension $r = 7$ with a MOR tolerance of 10^{-2}, where $\hat{A}$ and $\hat{E}$ are 7×7 matrices each, $\hat{B}$ is of 7×2, and $\hat{C}$ is of 3×7 matrices, respectively.

As shown in Section 5, we use Algorithm 1 to compute the low-rank Cholesky factors R_i and L_i. It has been observed in our tests that both the solutions R_i and L_i converge very well after 8 iterations. In Figure 1a we depict the normalized residual norms of R_i and L_i computed using relation (43), which reach very close to the level of 10^{-10} after the 8th iteration step.

In Figure 1b, we have depicted all the finite HSVs from the original and reduced systems. We notice that 9 of the finite 16 HSVs were truncated with MOR tolerance 10^{-2} for the reduced system and all the remaining HSVs exactly match with those of the original ones.

Since our model problem is a MIMO problem. the transfer function is matrix-valued. Therefore, we computed the $\mathcal{H}_2$ norm [4] of the transfer functions of the original and the reduced systems; instead of plotting component-wise transfer functions. In Figure 2a, we compare the $\mathcal{H}_2$ norms of the transfer functions of both systems to illustrate the efficiency of our MOR approach.

We observe that they overlap each other in the frequency range of $[10^2, 10^8]$ with a negligible error.

In Figure 2b, we show the absolute error norm between the transfer functions and the error bound computed using relation (11), from which it can be verified that the MOR approach approximates an efficient and accurate reduced model of the original system.

6.2 | Example-2: Artificial Data

In the second example, we construct our model problem from a continuous time model taken from section 4.3 of [31], where a spring-damper model is considered as an artificial model of piezo-mechanical systems. The original continuous-time model is converted here to a discrete-time model. The converted discrete-time model is an index-2 model with $n_1 = 10000$ and

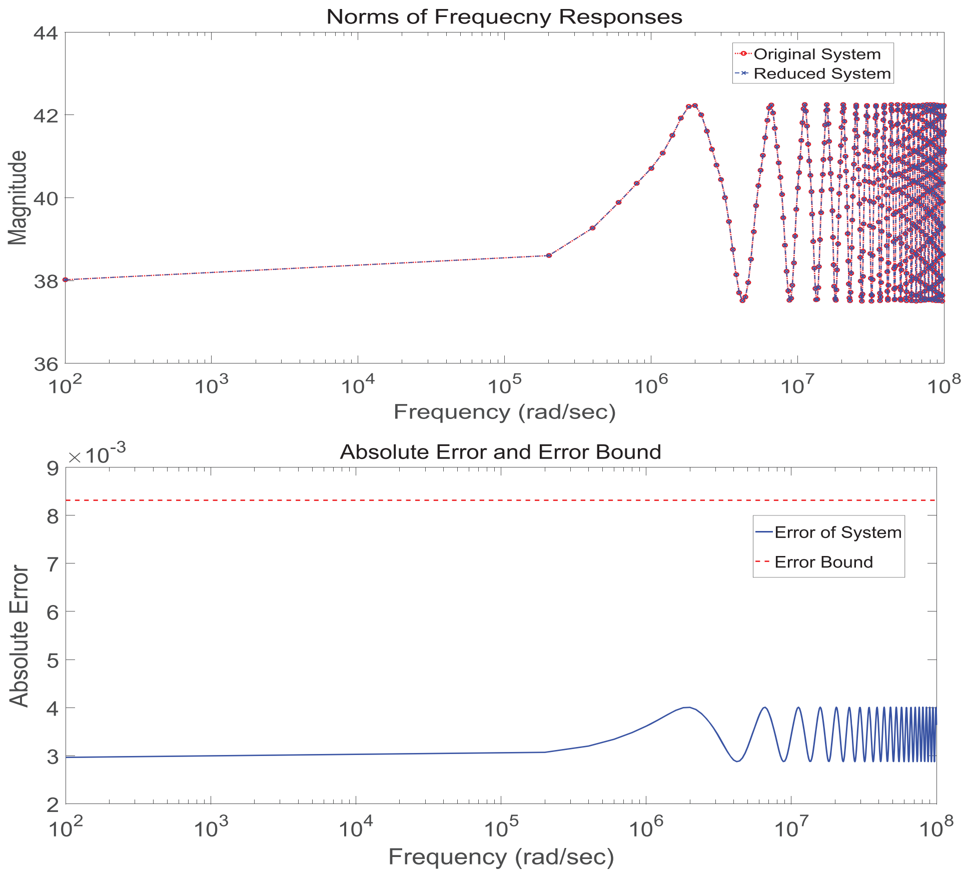

FIGURE 2 | (a) Transfer functions of original and reduced systems (up), (b) Error of original and reduced transfer functions (down).

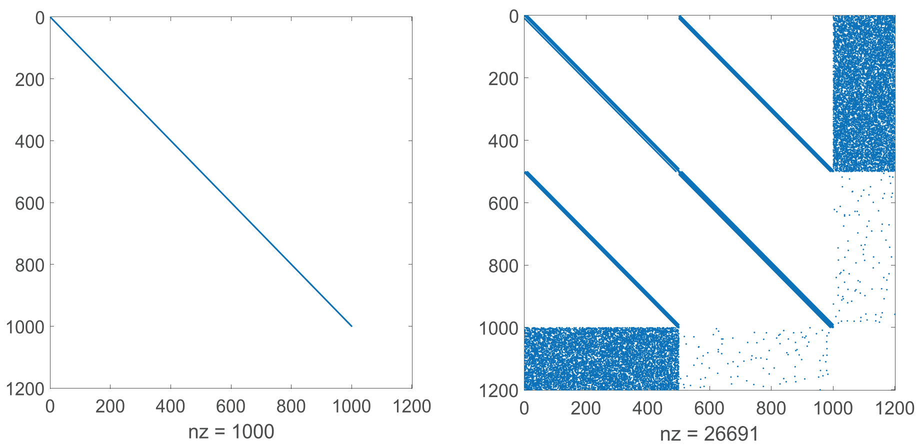

FIGURE 3 | Sparsity patterns of the discretized E (left) and A (right).

$n_2 = 2000$. Original system's dimension $n = n_1 + n_2 = 12000$. The MATLAB codes for this model formulation can be found in Appendix A.

The reduced system has the dimension $r = 11$ with MOR truncation tolerance, tol. $= 10^{-5}$.

The sparsity patterns of the discretized E and A matrices are shown in Figure 3.

In Figure 4a we have plotted the normalized residual norms of the approximated R_i and L_i computed by relation (43). We observe that they reach to the level of 10^{-14} after 20 iteration steps.

In Figure 4b we have plotted the largest 40 finite HSVs of the original system and the approximated 11 HSVs for the reduced system.

Similar to the first example, we compare the norms of the frequency responses of both the original and the reduced-order systems in Figure 5a. We also depict the absolute error in the approximation of the frequency response in Figure 5b. Based on these two figures, we conclude that The reduced order model (ROM) generates a good approximation of the original system.

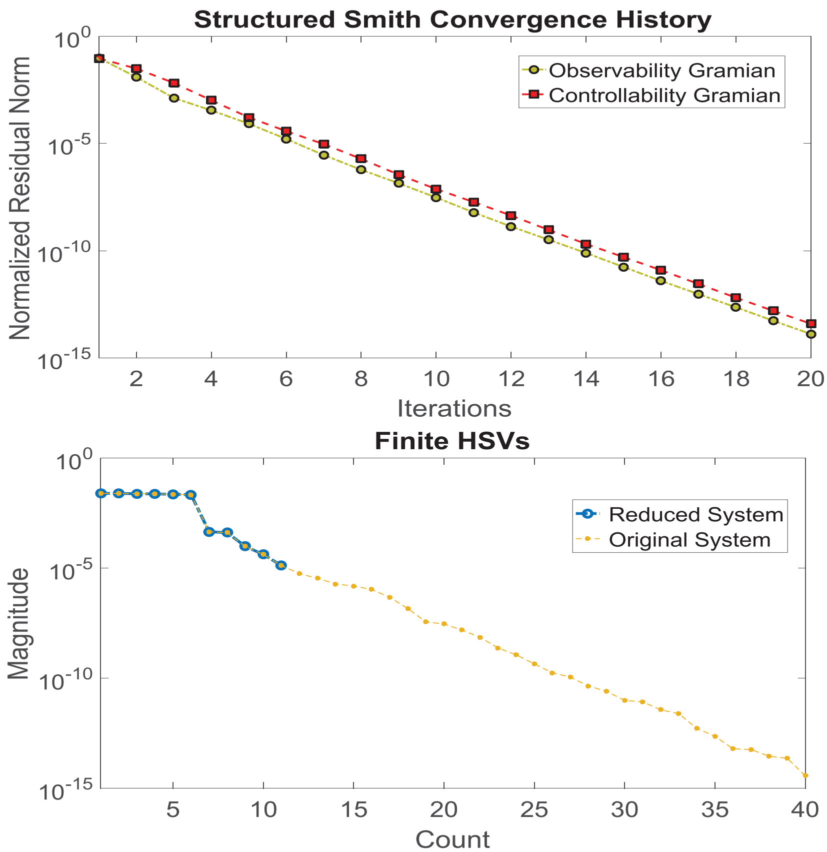

FIGURE 4 | (a) Normalized residual norms (up), (b) HSVs of original and reduced systems (down).

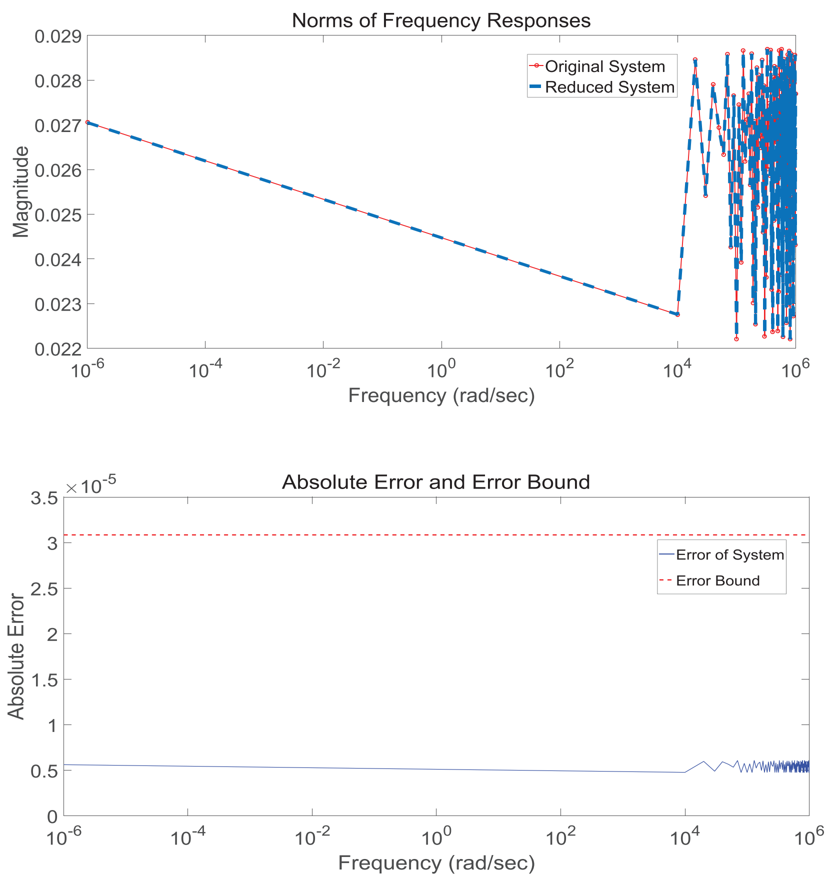

FIGURE 5 | (a) Transfer functions (up) (b) Error of original and reduced transfer functions (down).

TABLE 1 | ROMs with different truncation tolerances.

Dimension of original model	Truncation tolerance	Dimension of ROM
12000	10^{-6}	16
	10^{-5}	11
	10^{-4}	8
	10^{-3}	6

We would like to mention that the dimensions of the reduced order model can be changed by changing the tolerance levels of truncation. In Table 1 we have shown the dimensions of the computed reduced order models for different tolerance levels of MOR truncation.

7 | Conclusion

This paper presents a model reduction method for DT-LTI index-2 descriptor systems based on BT. In our approach, we reformulate the system in an equivalent ODE realization by embedding the non-causal subsystem onto the causal subsystem. However, after the reformation, the sparse matrices of the original system become dense. As a result, the computational cost of the BT-MOR method increases significantly. Therefore, we have developed a way to apply the MOR techniques directly on the original system's matrices without computing the projectors explicitly. The developed MOR algorithm for the discrete-time descriptor system is reformulated from an existing MOR strategy proposed for the continuous-time system. The results presented in this paper show good convergence of the approximated solutions and nice matching of the approximated transfer function with the original one. It also presents that the error generated by the MOR process is bounded.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

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Appendix A

The MATLAB codes of the index-2 model problem used in Example 2. The artificial data is prepared by considering the most general case of an index-2 discrete-time model having the specific sparse structure of the system (3). The following codes will construct the matrices E, A, B, and C for an index-2 system (3). Please note that we have fixed the random seed to generate the random matrices K_{up} and A2 in the following example.

```
----------------------------------------------------
This is an index 2 problem
----------------------------------------------------
nf=5000; n2=2000; % order of the system:
n=2*nf+n2=12000
nin=3; % number of input
nout=2; % number of output
i=3;
I=speye(nf);
n_p=2*nf-n2;

M=I+spdiags(i*ones(nf,1),i,nf,nf)
  +spdiags(-i*i*ones(nf,1),i,nf,nf)
  +spdiags(i*ones(nf,1),-i*i,nf,nf)
  +spdiags(-i*ones(nf,1),i*i,nf,nf);

K_uu=spdiags(i*ones(nf,1),-i*i,nf,nf)
     +spdiags(-i*ones(nf,1),-i*i,nf,nf)
     +spdiags(-i*ones(nf,1),i*i,nf,nf)
     +spdiags(-i*i*ones(nf,1),i,nf,nf)
     +spdiags(i*i*ones(nf,1),-i,nf,nf);

 mu= 0.5+.1*i;
 nu= 0.8+.1*i;
 D= mu*M+nu*K_uu;
 den= 0.02;

s1=rng      % define random seed for K_up
K_up= sprand(nf,n2,0.1*den);
rng(s1)

   % Construct an index-2 system %

A1=0.05*[M K_uu'; K_uu D];
s2=rng      % define random seed for A2
A2=0.05*[sprand(nf,n2,0.1); K_up];
rng(s2)

A3=A2'; A4=[spalloc(n2,n2,0)];

B1=[spalloc(nf,nin,0); spdiags(ones(nf,1),0,nf,nin)];
B2=spdiags(zeros(n2,1),0,n2,nin);

C1=[spdiags(ones(nf,1),0,nout,nf) spalloc(nout,nf,0)];
C2=spdiags(zeros(n2,1), 0, nout, n2);

E= [E1 spalloc(size(A2,1),size(A2,2),0);
spalloc(size(A3,1),size(A3,2),0)

spalloc(size(A4,1),size(A4,2),0)];

A=[A1 A2; A3 A4];
B=[B1; B2];
C=[C1 C2];
```